18-819F: Introduction to Quantum Computing 47-779/785: Quantum Integer Programming & Quantum Machine Learning

Quantum Fourier Transform

Lecture 15

2021.10.26

Agenda

- Primitive roots of unity (one)
  - Roots of unity on the real number line $\mathbb{R}$
  - Roots of unity in the complex number field $\mathbb{C}$
  - Discrete Fourier transform

- Quantum Fourier transform
  - Circuit representation of the QFT

Primitive Roots of Unity

- In conventional algebra, the equation $x^n - 1 = 0$ or $x^n = 1$ has one solution, $x = 1$ when n is odd, and two solutions, $x = \pm 1$, when it is even. This is when x is restricted to be real.

- When dealing with complex numbers, a similar expression for a complex number z,

  $z^n = 1$ Eqn. (15.1) has many roots.

- Using complex polar notation for z, we can write: $z = re^{j\vartheta} \Longrightarrow z^n = r^n e^{jn\vartheta} = 1$ Eqn. (15.2).

- Eqn. (15.2) is equivalent to $r^n e^{jn\vartheta} = 1e^{j2\pi k}$ for $k = 0, 1, 2, \ldots n-1$. This only makes sense if $r^n = 1$ and $\vartheta = 2\pi k/n$ (the complex number $j = \sqrt{-1}$).

- The roots of $z^n = 1$ can therefore be listed as $e^{j2\pi 0/n}, e^{j2\pi 1/n}, e^{j2\pi 2/n}, \ldots e^{j2\pi(n-1)/n}$.

- First root of unity ($n = 1, k = 0$) is 1; second roots of unit ($n = 2, k = 0, k = 1$) are 1 and -1; third roots of unity ($n = 3, k = 0, k = 1, k = 2$) are 1, $e^{j2\pi/3}, e^{j4\pi/3}$; and the fourth roots of unity are $1, j, -1, -j; etc.;$

Roots of Unity and the Discrete Fourier Transform

- The roots of unity can be written as powers of $\omega = e^{j2\pi/n}$.

- We now use the result above to define the Fourier transform, commonly used in signal processing in electrical and computer engineering, physics, computer science, and other disciplines.

- In signal processing, the Fourier transform is a tool that tells us what frequencies are contained in a time-dependent signal and how much of each frequency.

- In a discrete setting, the discrete Fourier transform (DFT) is an invertible matrix D of dimension $N,$ given as

$$D_{kl} = \frac{1}{\sqrt{N}}\omega^{kl} \quad \text{Eqn. (15.3), where } \omega = e^{j2\pi/N}.$$

- The DFT takes a series of N complex numbers, $x_0, x_1, \ldots x_{N-1}$ and transforms them into another series of complex numbers $y_0, y_1, \ldots y_{N-1}$.

Discrete Fourier Transform

- The definition in Eqn. (15.3) means that for a number x_n, we can obtain its discrete Fourier transform (DFT), y_k as

$$y_k \longrightarrow \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e^{\frac{2\pi j}{N}kn} x_n \quad \text{Eqn. (15.4).}$$

- The inverse discrete Fourier transform $(\mathrm{DFT}^{\dagger})$ is given as

$$x_n \longrightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{-2\pi j}{N}kn} y_k \quad \text{Eqn. (15.5).}$$

- Note that the Fourier basis states are orthogonal and the DFT expressions in (15.4) and (15.5) are inverses of each other. From now on we may set $N = 2^n$.

Quantum Fourier Transform

- We define the quantum Fourier transform (QFT) in an analogous way to the the DFT such that for an orthonormal basis $|x\rangle \in \{|0\rangle, |1\rangle, \ldots |N-1\rangle\}$ the quantum Fourier transform is given as

$$|x\rangle \longrightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \omega^{x.k} |k\rangle \text{ Eqn. } (15.6)$$

- The inverse quantum Fourier transform is the given as

$$|k\rangle \longrightarrow \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} \omega^{-x.k} |x\rangle \text{ Eqn. } (15.7)$$

- The integer k in binary notation is

$$k = \sum_{\ell=0}^{n} k_\ell 2^{n-\ell}$$

- For Fourier basis states $|i\rangle$, we can write $|i\rangle$ using k as expanded above,

$$|i\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp\left(j2\pi i \sum_{\ell=0}^{n} k_\ell 2^{-\ell} \right) |k\rangle \text{ Eqn. } (15.8)$$

Expansion of Quantum Fourier Transform Expression

- Expression (15.8) can be expanded by writing the sum in the exponential as (tensor) products, thus

$$|i\rangle = \frac{1}{\sqrt{N}} \sum_{k_1,k_2,\ldots k_n} \exp(j2\pi i k_1 2^{-1}) \exp(j2\pi i k_2 2^{-2}) \ldots \exp(j2\pi i k_n 2^{-n}) \, |k_1 \ldots k_n\rangle \quad \text{Eqn. (15.9)}$$

- Since each summation term k_n is $\{0,1\}$, the expression in (15.9) becomes

$$|i\rangle = \frac{1}{\sqrt{N}} (|0\rangle + \exp(j2\pi i) \, 2^{-1}|1\rangle)(|0\rangle + \exp(j2\pi i) \, 2^{-2}|1\rangle) \ldots (|0\rangle + \exp(j2\pi i) \, 2^{-n}|1\rangle) \quad \text{Eqn. (15.10).}$$

- The binary fraction $\frac{i}{2^m}$ can be represented as $\frac{i}{2^m} = 0.i_{n-m+1} \ldots i_n$, which is a decimal expansion of i up to m bits.

- The state in Eqn. (15.10) can there before be rewritten as

$$|i\rangle = \frac{1}{\sqrt{N}} (|0\rangle + \exp(j2\pi 0.i_n) \, |1\rangle)(|0\rangle + \exp(j2\pi 0.i_{n-1} i_n) \, |1\rangle) \ldots (|0\rangle + \exp(j2\pi 0.i_1 \ldots i_n) \, |1\rangle) \quad \text{Eqn. (15.11)}$$

Another Perspective of the Quantum Fourier Transform

- Another way to write Eqn. (15.10) using tensor notation is

$$|i\rangle = \frac{1}{\sqrt{N}}(|0\rangle + \exp(j2\pi 0.i_n)\,|1\rangle) \otimes (|0\rangle + \exp(j2\pi 0.i_{n-1}i_n)\,|1\rangle) \otimes \cdots \otimes (|0\rangle + \exp(j2\pi 0.i_1 \ldots i_n)\,|1\rangle) \quad \text{Eqn. (15.11)}$$

- Notice that the last qubit (in red) depends on all the input qubits. The other qubits toward the left depend less on the input qubits.

- When a Hadamard is applied to the first of the input qubits $|i_1 i_2 i_3 \ldots i_n\rangle$, we get

$$|i\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi j 0.i_1}\right)|i_2 i_3 \ldots i_n\rangle$$

- We learned earlier that a rotation gate is: $R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi j/2^k} \end{pmatrix}$; applying controlled rotation gates $R_2, R_3,$ etc., to the appropriate qubits, eventually get all terms of the input qubit into the phase term. We explain the action of the controlled rotation later, but in the meanwhile, this is what we want in the state.

$$|i\rangle \longrightarrow \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi j 0.i_1 i_2 i_3 \ldots i_n}\right)|i_2 i_3 \ldots i_n\rangle.$$

Action of the Controlled Rotation Operator

- We have the rotation operator as

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi j/2^k} \end{pmatrix}.$$

- When this operator acts in controlled fashion on a two-qubit state $|x_k x_\ell\rangle$ where the first qubit is the control and the second is the target, we get

$$R_k|0x_\ell\rangle = |0x_\ell\rangle$$

$$R_k|1x_\ell\rangle = \exp\left(\frac{2\pi j}{2^k} x_k\right)|1x_\ell\rangle.$$

- The controlled rotation operator is used for importing the input to the exponential as we show next.

- The controlled rotation operator, together with the Hadamard operator can be used to synthesize a circuit for the quantum Fourier transform.

Another Perspective of the Quantum Fourier Transform

- Continuing the process of importing the qubits into the phase by applying a Hadamard and controlled rotations eventually get us to

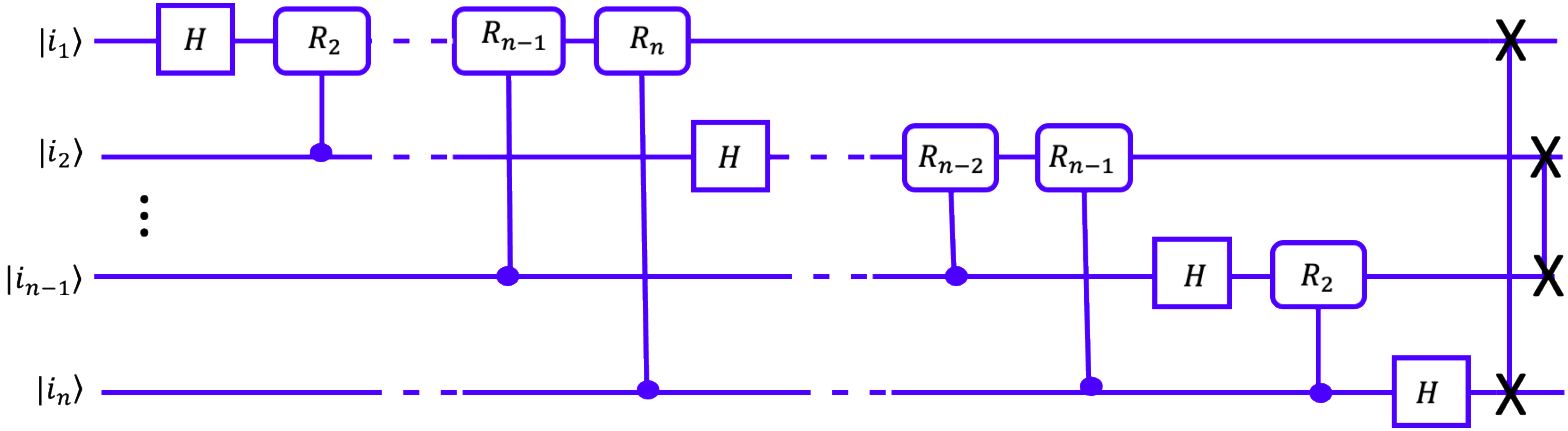
$$|i\rangle = \frac{1}{\sqrt{2^n}}\left(|0\rangle + e^{2\pi j 0.i_1 i_2 \ldots i_n}|1\rangle\right) \otimes \left(|0\rangle + e^{2\pi j 0.i_1 i_2 \ldots i_{n-1}}|1\rangle\right) \otimes \cdots \otimes \left(|0\rangle + e^{2\pi j 0.i_n}|1\rangle\right)$$

- This is the desired full QFT.

- Depending on the order in which the qubits were entered, or which order the Hadamard operator acts on the qubits, it maybe necessary to perform a swap operation at the end of the QTF process.

- A typical circuit implementation of the QFT is shown on the next slide.

Gate implementation of the QFT



What Does a Quantum Fourier Transform Do?

- The quantum Fourier transform (QFT) takes a state vector, $|\psi\rangle$, given as

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \cdots + \alpha_{N-1}|N-1\rangle \text{ Eqn. (15.12),}$$

and transforms it into another state vector, $|\Psi\rangle$, given by

$$|\Psi\rangle = \beta_0|0\rangle + \beta_1|1\rangle + \cdots + \beta_{N-1}|N-1\rangle \text{ Eqn. (15.13).}$$

- The importance of the QFT comes from its ability to perform a transformation on superpositions of states. For example, we can perform a QFT on the state

$$|\varphi\rangle = \frac{1}{\sqrt{4}}(|00\rangle + |01\rangle - |10\rangle - |11\rangle) \quad \text{Eqn. (15.14) or even on the state, } |\chi\rangle$$

$$|\chi\rangle = \frac{1}{\sqrt{3}}(|1\rangle + |3\rangle - |7\rangle) = \frac{1}{\sqrt{3}}(|001\rangle + |011\rangle - |111\rangle) \text{ Eqn. (15.15).}$$

- In Eqn. (15.15), most of the basis states have amplitudes that are zero.

Action of Quantum Fourier Transform on State Vector

- For a state vector given as linear superposition

$$|\varphi\rangle = \sum_{k=0}^{2^n-1} \alpha_k |k\rangle \quad \text{Eqn. (15.16)}$$

- One defines the QFT as

$$|\Phi\rangle = QFT|\varphi\rangle = \sum_{k=0}^{2^n-1} \sum_{\ell=0}^{2^n-1} \frac{\alpha_k e^{2\pi jk\ell/2^n}}{\sqrt{2^n}} |\ell\rangle$$

- One can identify a matrix $N_{k\ell} = e^{2\pi jk\ell/2^n}$ so that the Fourier transform above can be rewritten as

$$|\Phi\rangle = QFT|\varphi\rangle = \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} \left(\sum_{\ell=0}^{2^n-1} \alpha_k N_{k\ell} \right) |\ell\rangle$$

- One can evaluate the matrix elements for a few values of k and ℓ.

Matrix Elements of the QFT

- The matrix elements $N_{k\ell} = e^{2\pi jk\ell/2^n}$ of the QFT for a few values of k and ℓ are

$$N_{k\ell} = \frac{1}{\sqrt{2^n}}\begin{bmatrix} e^{2\pi j0.0/2^n} = e^0 = 1 & e^{2\pi j1.0/2^n} = e^0 = 1 & \dots \\ e^{2\pi j0.1/2^n} = e^0 = 1 & e^{2\pi j1.1/2^n} = \omega & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \quad \text{Eqn. (15.17)}$$

- For one qubit, we have $n = 1$, and the matrix becomes $N = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & e^{j\pi} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ Eqn. (15.18)

- Evidently the matrix for the QFT for a single qubit is identical to the Hadamard operator, H.

Determining a QFT of a Single Qubit

- Determine the QFT of the qubit

$$|\psi\rangle = \frac{1}{\sqrt{4}}|0\rangle + \sqrt{\frac{3}{4}}|1\rangle = \begin{pmatrix} 1/\sqrt{4} \\ \sqrt{3/4} \end{pmatrix}.$$

- The matrix representation for a single qubit is

$$N = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & e^{j\pi} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

- The QFT: $|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{pmatrix} 1/\sqrt{4} \\ \sqrt{3/4} \end{pmatrix} = \begin{pmatrix} (1+\sqrt{3})/2\sqrt{2} \\ (1-\sqrt{3})/2\sqrt{2} \end{pmatrix}.$

- The transformed qubit can therefore be rewritten as

$$|\Psi\rangle = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{3})|0\rangle + (1-\sqrt{3})|1\rangle\right].$$

Two Qubit Quantum Fourier Transform

- Given the two-qubit state $|\varphi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$ with $n = 2$, the matrix representation of the QFT operator becomes

$$N_{k\ell} = \omega^{k\ell} = e^{2\pi jk\ell/2^n} = e^{2\pi jk\ell/4}$$

and the resulting 4×4 matrix is

$$N_{k\ell} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{\pi j/4} & e^{\pi j2/4} & e^{\pi j3/4} \\ 1 & e^{\pi j2/4} & e^{\pi j4/4} & e^{\pi j6/4} \\ 1 & e^{\pi j3/4} & e^{\pi j6/4} & e^{\pi j9/4} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

- Since $|\varphi\rangle$ can be written as the bra $\langle\varphi| = 1/\sqrt{2}\ (1\ 0\ 1\ 0)$ the QFT is calculated from

$$QFT\colon\ |\varphi\rangle = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{\pi j/4} & e^{\pi j2/4} & e^{\pi j3/4} \\ 1 & e^{\pi j2/4} & e^{\pi j4/4} & e^{\pi j6/4} \\ 1 & e^{\pi j3/4} & e^{\pi j6/4} & e^{\pi j9/4} \end{bmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{pmatrix}$$

Generalized Matrix for the Quantum Fourier Transform

- Eqn. (15.17) provided a way to generate the matrix elements of the quantum Fourier transform. If we continue the calculation indicated in that equation, we arrive at the general matrix of the form below, where we have reverted to using the notation for the primitive roots of unity, $\omega^{k\ell}$.

$$QFT_N = \frac{1}{\sqrt{N}}\begin{pmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \dots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \dots & \omega^{2N-2} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \dots & \omega^{3N-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2N-2} & \omega^{3N-3} & \dots & \omega^{(N-1)(N-1)} \end{pmatrix} \quad \text{Eqn. (15.19).}$$

- Another way of writing the $k\ell - th$ entry the QFT_N matrix is simply $\omega^{k\ell}$.

Properties of the Quantum Fourier Transform

- **First Property**: the QFT is unitary – an operator is unitary if its columns are orthonormal; to prove this, we simply multiply any two columns of the QFT matrix and multiply them. We take the C_k and C_ℓ columns, then

$$\langle C_k|C_\ell\rangle = \frac{1}{N}\sum_{n=0}^{N-1} \omega^{nk}\left(\omega^{n\ell}\right)^\dagger = \frac{1}{N}\sum_{n=0}^{N-1}\left(\omega^{k-\ell}\right)^n = 1 \text{ for } k = \ell$$

- When $k \neq \ell$, the expression above is a geometric series.

- And $\langle C_k|C_\ell\rangle = \frac{1}{N}\sum_{n=0}^{N-1}\left(\omega^{k-\ell}\right)^n = \frac{1}{N}\frac{\omega^{N(k-\ell)}-1}{\omega^{k-\ell}-1} = 0$ because $\omega^{N(k-\ell)} = 1$ due the fact that ω is the Nth root of unity.

Properties of the Quantum Fourier Transform

- **Second Property:** Linear Shift - if the function $|f(t)\rangle$ has a Fourier transform $|F(t)\rangle$ then $|f(t+\tau)\rangle$ has a Fourier transform $|F(t)\rangle e^{2\pi t\tau/N}$.

- Linear shift is an important property for quantum measurements. We consider this in the following. Given the state vectors

$$|k\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ and } |\ell\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$ we calculate their QFT by using the matrix below

$$QFT_4 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}$$

- The results are $QFT_4|k\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ and $QFT_4|\ell\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix}$.

Properties of the Quantum Fourier Transform

- Our results in determining the QFT for the state vectors $|k\rangle$ and $|\ell\rangle$ indicate that except for a relative phase difference, the magnitudes of the state vectors are exactly the same when a measurement operation is performed. Recall that we learned a phase iii quantum mechanics has no physical significance.

- This means the QFT is an important operator for distinguishing quantum states. If the measurement results of a computation were

$\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ and $\frac{1}{2}\begin{pmatrix}1\\j\\-1\\-j\end{pmatrix}$, applying a QFT the result would allow us to infer the that original states were

$|k\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ and $|\ell\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$. The QFT is therefore a critical operator in most quantum algorithms.

Symmetry and Superposition in the QFT

- The ordinary discrete Fourier transform (DFT) has a version called the fast Fourier transform (FFT). It owes it fast determination to symmetry. Since the DFT is an $N \times N$ matrix, it pays to study the elements of the matrix and see if there is symmetry in them. Below is a 6×6 matrix we will use for discussions. The exponents are written modulo 6. Notice that column 2 is similar to column 6 and row 2 is similar to row 6. This observation can be used to split the columns into even and odd columns.

$$QFT_{6\times 6} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 \\ 1 & \omega^2 & \omega^4 & 1 & \omega^2 & \omega^4 \\ 1 & \omega^3 & 1 & \omega^3 & 1 & \omega^3 \\ 1 & \omega^4 & \omega^2 & 1 & \omega^4 & \omega^2 \\ 1 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{pmatrix}$$

- Straight-forward multiplication in determining the DFT of a function would lead to $\mathcal{O}(N^2)$ steps. By levering symmetry, the FFT reduces the number of steps to $\mathcal{O}(N log_2 N)$.
- The symmetry exploited is to divide the FFT into transforms of size $N/2$ odd and even terms and to continue reducing the size by half again and again until one is performing simple calculations fast.

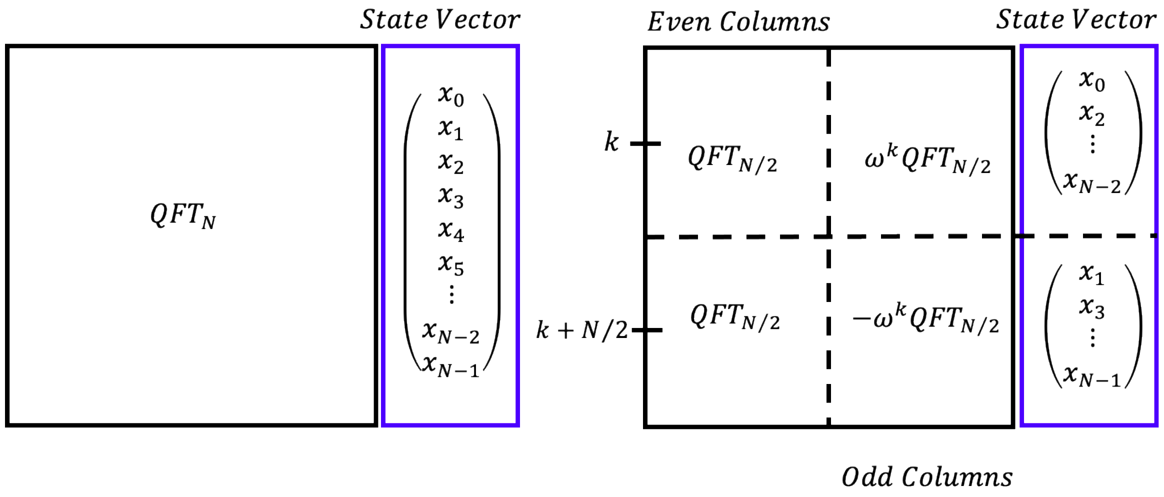
Symmetry in the Quantum Fourier Transform

- One can exploit the same symmetry in the calculation of the QFT. First, divide the QFT_N into transforms of size $QFT_{N/2}$, then $QFT_{N/4}$ and so on until one is performing N calculations of QFT_1, which are faster to do.

- The process of sub-dividing the QFT into smaller chunks is illustrated in the graphic on the rigth.

- For the case considered, we have assumed that $N = 2^n$.

- We also note that $\omega^{k+N/2} = -\omega^k$ and $\omega^{k+N} = \omega^k$.



Intuition Behind the Quantum Fourier Transform

- The QFT allows us to change the basis from the computational basis (often called the Z-basis), which in our case has been $|0\rangle$ and $|1\rangle$, to the Fourier basis or vice versa. We can represent the QFT as below:

- $|\text{State in Computational Basis}\rangle \xrightarrow{\text{Quantum Fourier Transformation}} |\text{Fourier Basis}\rangle$

- In ordinary signal processing, Fourier transformation is simply going from the time domain to the frequency domain, where are determine the frequency content of a time domain signal. This is of critical importance it means we can filter out unwanted frequencies (noise) from the signal we desire.

- The QFT plays the same role in computation in that it provide methods for enhancing amplitudes of desirable state vectors while minimizing those we do not want.

Summary

- Introduced primitive roots of unity
  - Showed relationship to DFT
  - Established relationship of DFT to QFT

- Discussed gate model of QFT
  - Example of computing the QFT